COMPLETELY MONOTONE FUNCTIONS IN THE STUDY OF A CLASS OF FRACTIONAL EVOLUTION EQUATIONS

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Dedicated to the memory of Professor Mirjana Stojanović

Inspired by:

Mainardi, F., Mura, A., Gorenflo, R., Stojanović, M.
The two forms of fractional relaxation of distributed order,
J. Vib. Control. 13 (2007) pp. 1249–1268.

Gorenflo, R., Luchko, Yu., Stojanović, M.
Fundamental solution of a distributed order time-fractional diffusion-wave equation as probability density,
Fract. Calc. Appl. Anal. 16, No.2 (2013) pp. 297–316.

Completely monotone functions ($\mathcal{CMF}$) and Bernstein functions ($\mathcal{BF}$)

A function $f : (0, \infty) \to \mathbb{R}$ is called **completely monotone** if it is of class C^∞ and

$$(-1)^n f^{(n)}(t) \geq 0, \text{ for all } t > 0, \; n = 0, 1, ...$$

(The simplest example: $e^{-\lambda t}$, $\lambda > 0$.)

Bernstein's theorem: $f(t) \in \mathcal{CMF}$ if and only if

$$f(t) = \int_0^\infty e^{-tx} \, dg(x),$$

where $g(x)$ is nondecreasing and the integral converges for $0 < t < \infty$.

A C^∞ function $f : (0, \infty) \to \mathbb{R}$ is called a **Bernstein function** if

$$f(t) \geq 0 \text{ and } f'(t) \in \mathcal{CMF}.$$

Some useful properties

Proposition:

(a) The class $\mathcal{CMF}$ is closed under pointwise addition and multiplication;

(b) The class $\mathcal{BF}$ is closed under pointwise addition, but, in general not under multiplication;

(c) If $f \in \mathcal{CMF}$ and $\varphi \in \mathcal{BF}$, then the composite function $f(\varphi) \in \mathcal{CMF}$;

(d) If $f \in \mathcal{BF}$, then $f(t)/t \in \mathcal{CMF}$;

(e) Let $f \in L^1_{loc}(\mathbb{R}_+)$ be a nonnegative and nonincreasing function, such that $\lim_{t\to+\infty} f(t) = 0$. Then $\varphi(s) = s\widehat{f}(s) \in \mathcal{BF}$;

(f) If $f \in L^1_{loc}(\mathbb{R}_+)$ and $f \in \mathcal{CMF}$, then $\widehat{f}(s)$ admits analytic extension to the sector $|\arg s| < \pi$ and

$$|\arg \widehat{f}(s)| \leq |\arg s|, \quad |\arg s| < \pi.$$

The operators of fractional integration and differentiation

J_t^α - the Riemann-Liouville fractional integral of order $\alpha > 0$:

$$J_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

D_t^α - the Riemann-Liouville fractional derivative

${}^C D_t^\alpha$ - the Caputo fractional derivative

$$D_t^1 = {}^C D_t^1 = d/dt; \qquad {}^C D_t^\alpha = J_t^{1-\alpha} D_t^1, \quad D_t^\alpha = D_t^1 J_t^{1-\alpha}, \quad \alpha \in (0,1).$$

Mittag-Leffler function

Fractional relaxation equation ($\lambda > 0,\ 0 < \alpha \leq 1$):

$$ {}^{C}D_t^{\alpha} u(t) + \lambda u(t) = f(t), \quad t > 0, $$
$$ u(0) = c_0. $$

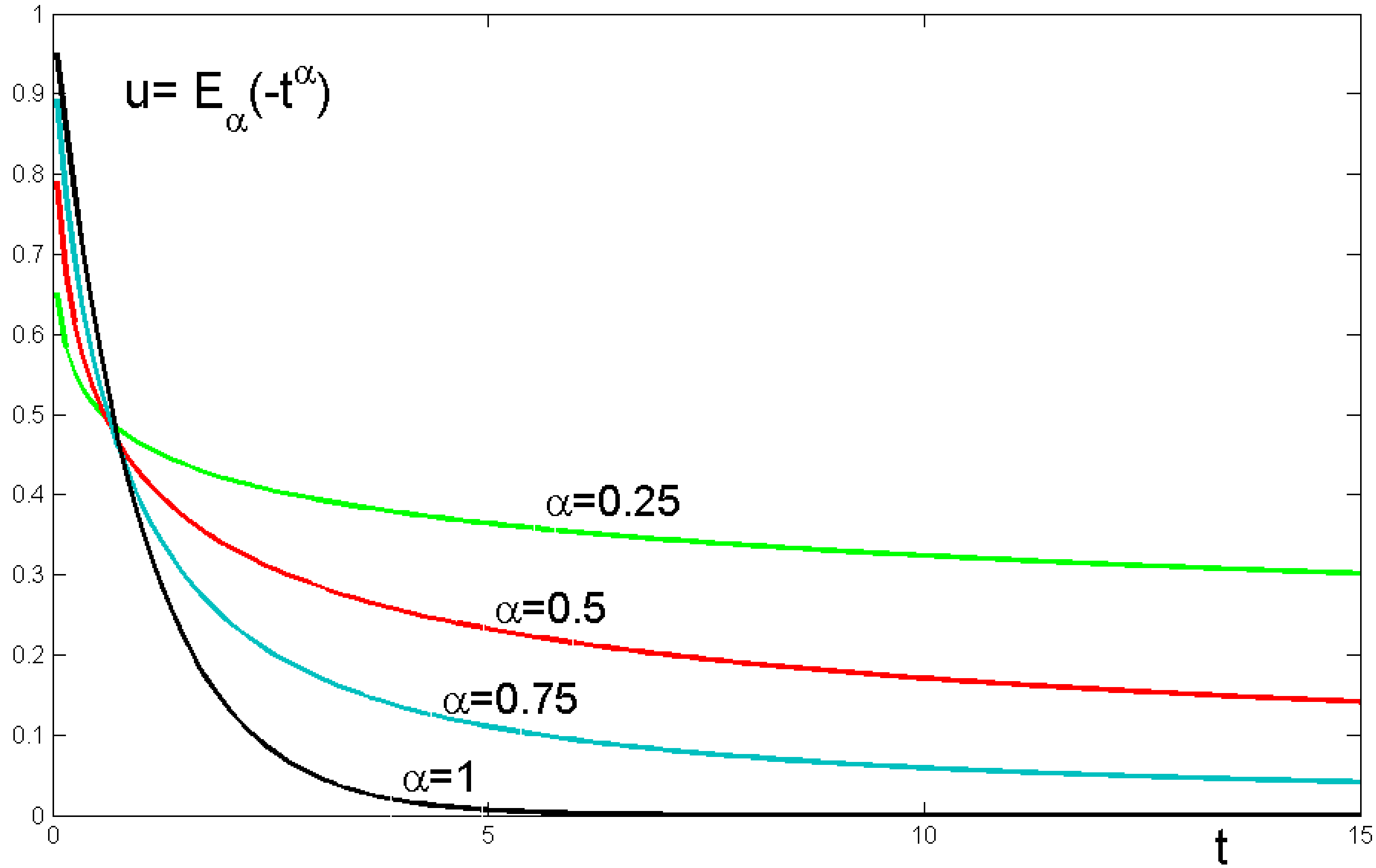
The solution is given by:

$$ u(t) = c_0 E_\alpha(-\lambda t^\alpha) + \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-\lambda \tau^\alpha) f(t-\tau)\, d\tau. $$

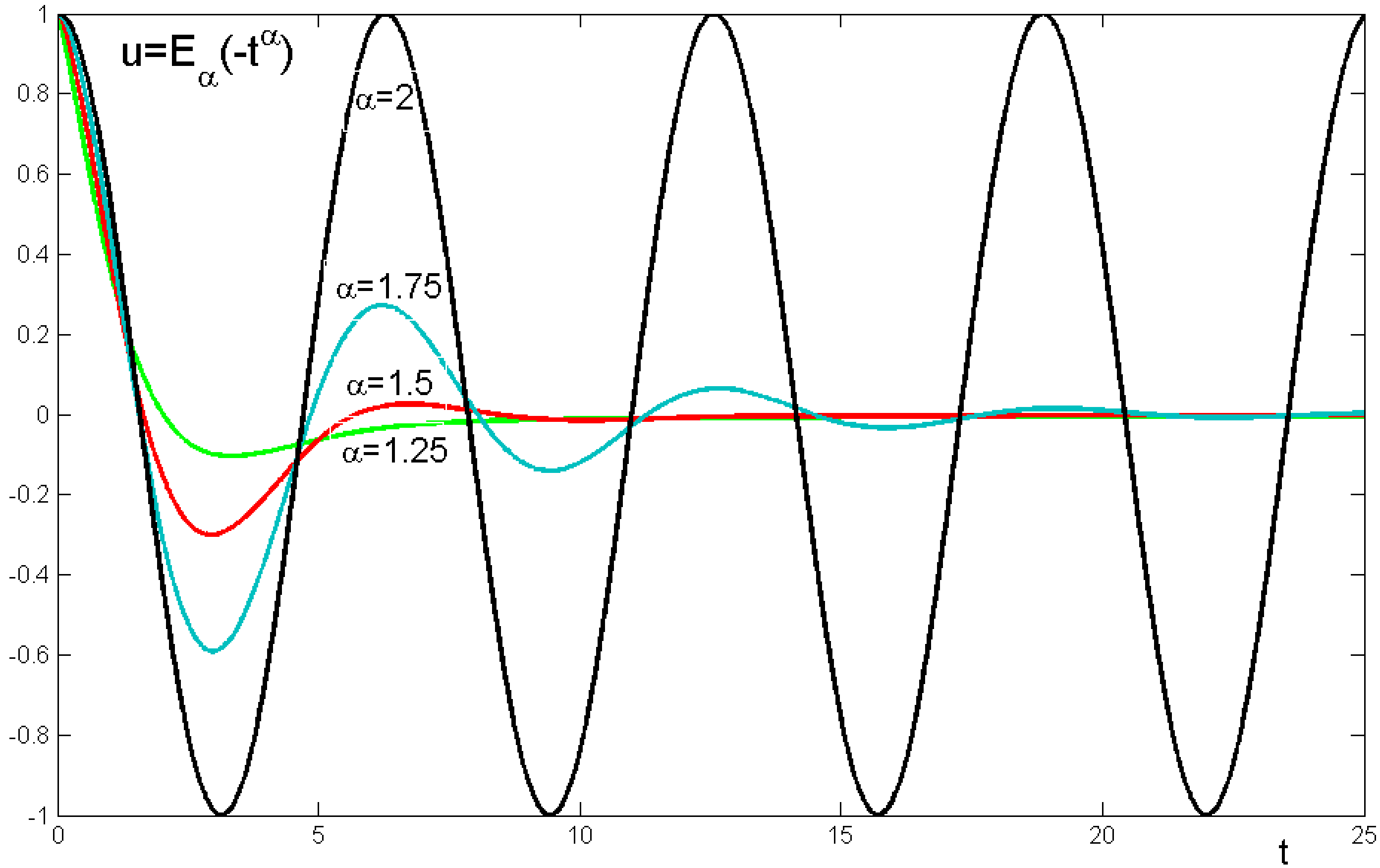
Mittag-Leffler function ($\alpha, \beta \in \mathbb{R},\ \alpha > 0$):

$$ E_{\alpha,\beta}(-t) = \sum_{k=0}^{\infty} \frac{(-t)^k}{\Gamma(\alpha k + \beta)}, \quad E_\alpha(-t) = E_{\alpha,1}(-t). $$

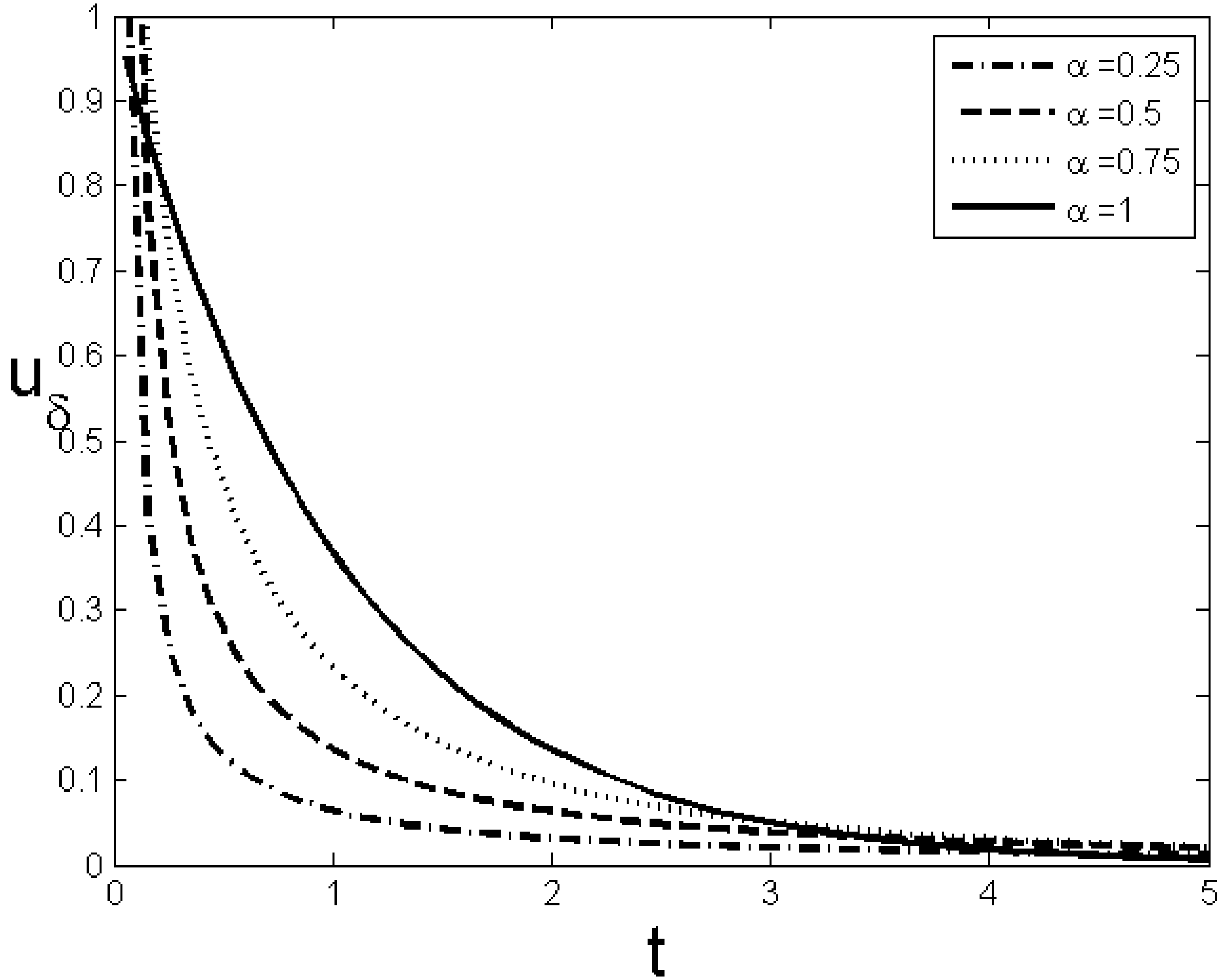
$E_1(-t) = e^{-t} \in \mathcal{CMF}$
$E_\alpha(-t) \in \mathcal{CMF}$, iff $0 < \alpha < 1$ (Pollard, 1948)
$E_{\alpha,\beta}(-t) \in \mathcal{CMF}$, iff $0 \leq \alpha \leq 1$, $\alpha \leq \beta$ (Schneider, 1996; Miller, 1999)



Plots of $E_\alpha(-t^\alpha)$ for different values of $\alpha \in (0, 1]$.
$\alpha = 1$ - exponential decay, $\alpha \in (0, 1)$ - algebraic decay ($t^{-\alpha}$).
Completely monotone functions.



Plots of $E_\alpha(-t^\alpha)$ for different values of $\alpha \in (1, 2]$.
No more complete monotonicity for $\alpha > 1$!
Damped oscillations.



Plots of $t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})$ for different values of $\alpha \in (0,1]$.
Completely monotone functions.

Fractional evolution equation of distributed order

Two alternative forms:

$$\int_0^1 \mu(\beta)\,{}^C D_t^\beta u(t)\,d\beta = Au(t), \quad t > 0, \tag{1}$$

and

$$u'(t) = \int_0^1 \mu(\beta) D_t^\beta Au(t)\,d\beta, \quad t > 0, \tag{2}$$

A - closed linear unbounded operator densely defined in a Banach space X

Initial condition: $u(0) = a \in X$.

Two cases for the weight function μ:

- discrete distribution

$$\mu(\beta) = \delta(\beta - \alpha) + \sum_{j=1}^{m} b_j \delta(\beta - \alpha_j), \tag{3}$$

where $1 > \alpha > \alpha_1 ... > \alpha_m > 0,\ b_j > 0,\ j = 1, ..., m,\ m \geq 0$, and δ is the Dirac delta function;

- continuous distribution

$$\mu \in C[0,1],\ \mu(\beta) \geq 0,\ \beta \in [0,1], \tag{4}$$

and $\mu(\beta) \neq 0$ on a set of a positive measure.

Discrete distribution:

Multi-term time-fractional equations in the Caputo sense

$$ {}^{C}D_t^{\alpha} u(t) + \sum_{j=1}^{m} b_j \, {}^{C}D_t^{\alpha_j} u(t) = Au(t), \quad t > 0, \tag{5} $$

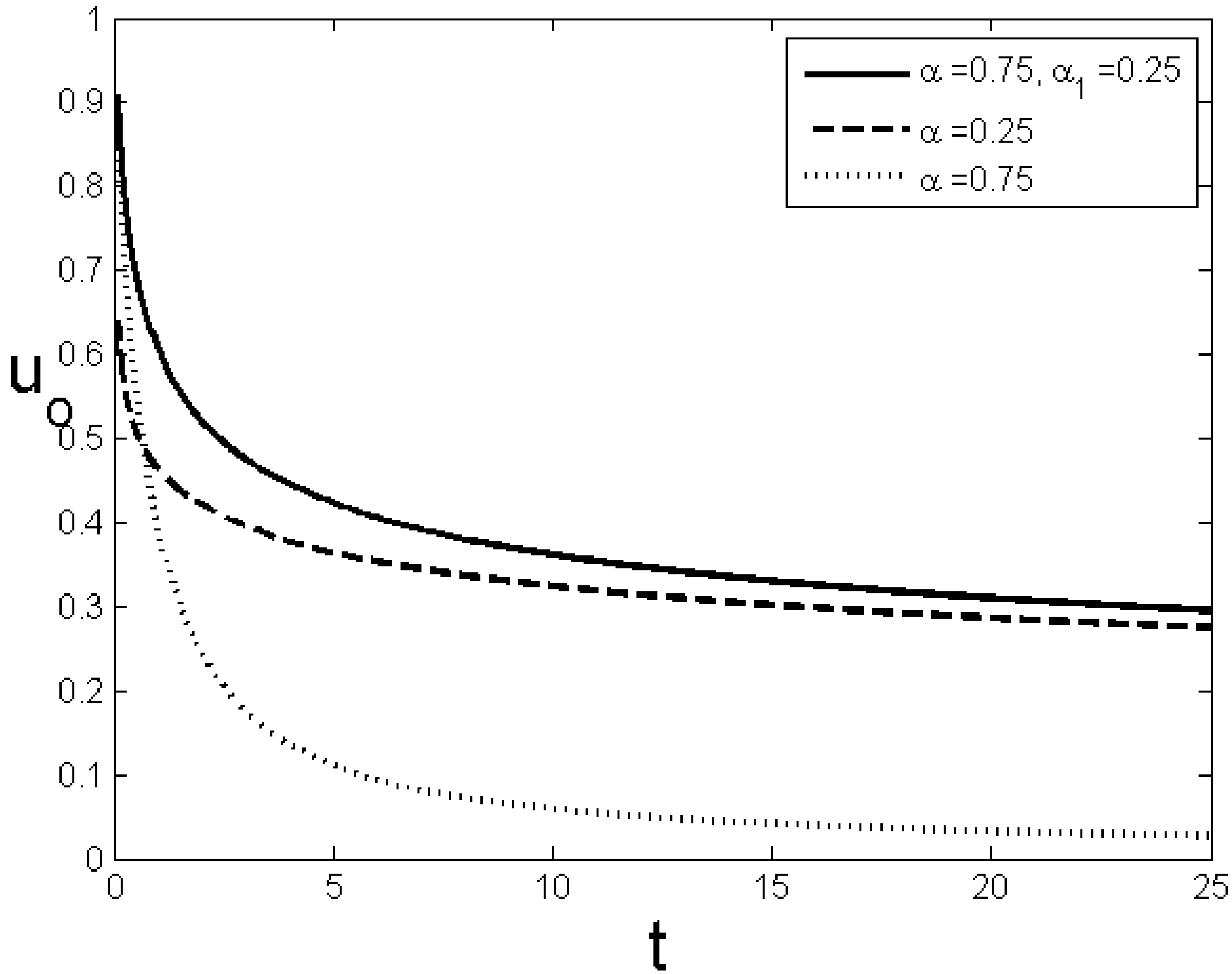
and in the Riemann-Liouville sense

$$ u'(t) = D_t^{\alpha} Au(t) + \sum_{j=1}^{m} b_j \, D_t^{\alpha_j} Au(t), \quad t > 0 \tag{6} $$

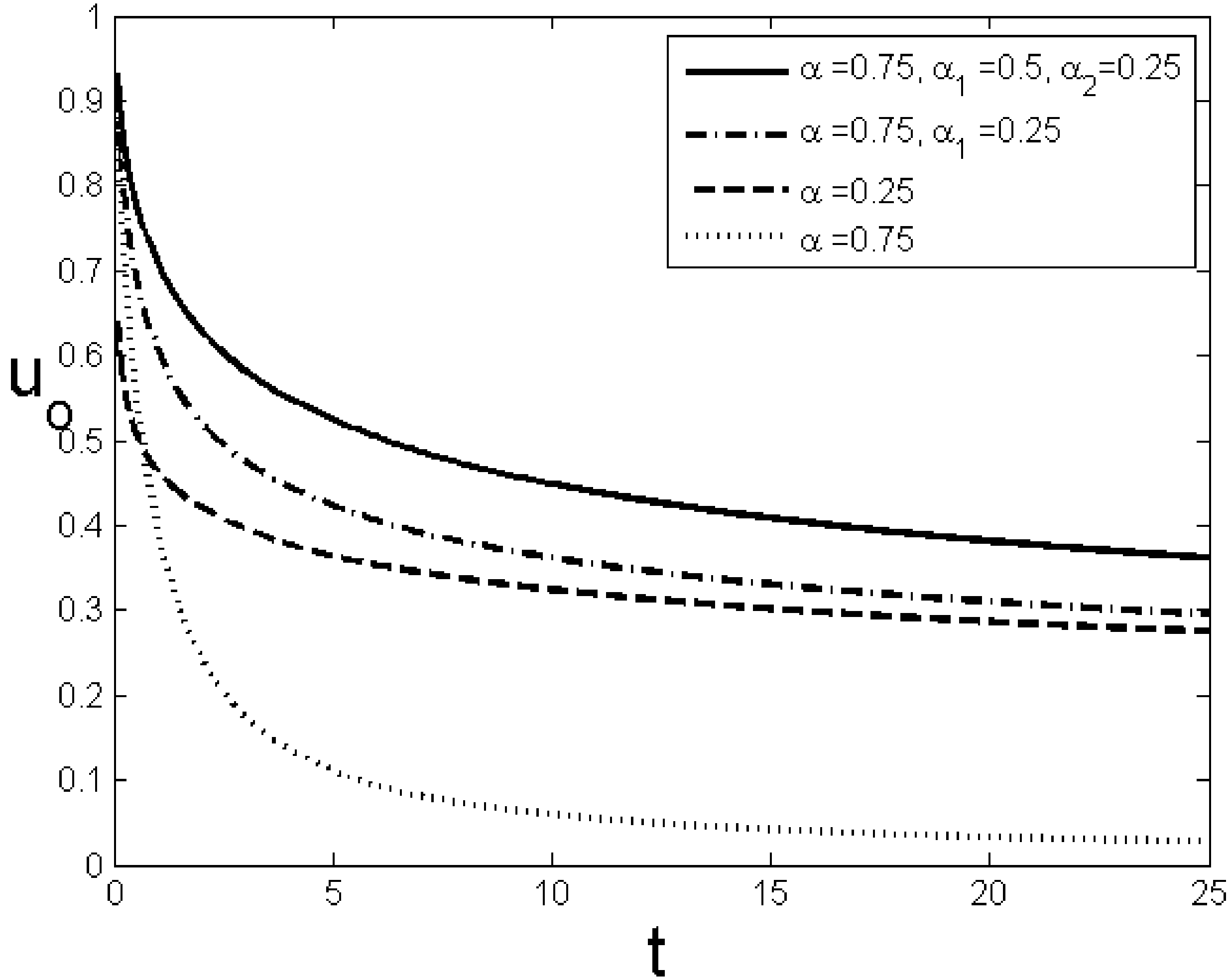
If $m = 0$ (single-term equations):
problem (5) is equivalent to (6) with α replaced by $1 - \alpha$.

All problems are generalizations of the **classical abstract Cauchy problem**

$$ u'(t) = Au(t), \quad t > 0; \qquad u(0) = a \in X. \tag{7} $$



Solution $u(t)$ of (5) with $A = -1$ for:
$m = 1$, $\alpha = 0.75$, $\alpha_1 = 0.25$,
$m = 0$, $\alpha = 0.25$
$m = 0$, $\alpha = 0.75$.



Solution $u(t)$ of (5) with $A = -1$ for:
$m = 2$, $\alpha = 0.75$, $\alpha_1 = 0.5$, $\alpha_2 = 0.25$
$m = 1$, $\alpha = 0.75$, $\alpha_1 = 0.25$,
$m = 0$, $\alpha = 0.25$
$m = 0$, $\alpha = 0.75$.

Unified approach to the four problems

Rewrite problems (1) and (2) as an abstract Volterra integral equation

$$u(t) = a + \int_0^t k(t-\tau)Au(\tau)\,d\tau, \quad t \geq 0; \qquad a \in X,$$

where

$$\widehat{k_1}(s) = (h(s))^{-1}, \; \widehat{k_2}(s) = h(s)/s,$$

In the continuous distribution case:

$$h(s) = \int_0^1 \mu(\beta)s^\beta\,d\beta.$$

In the discrete distribution case:

$$h(s) = s^\alpha + \sum_{j=1}^m b_j s^{\alpha_j}.$$

Define

$$g_i(s) = 1/\widehat{k_i}(s), \quad i = 1, 2.$$

Particular cases

In the single-term case:

$$k_1(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)},\ k_2(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}, \quad g_1(s) = s^\alpha,\ g_2(s) = s^{1-\alpha},$$

In the double-term case:

$$k_1(t) = t^{\alpha-1} E_{\alpha-\alpha_1,\alpha}(-b_1 t^{\alpha-\alpha_1}),\ k_2(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + b_1 \frac{t^{-\alpha_1}}{\Gamma(1-\alpha_1)},$$

$$g_1(s) = s^\alpha + b_1 s^{\alpha_1},\ g_2(s) = \frac{s}{s^\alpha + b_1 s^{\alpha_1}} = s\widehat{k_1}(s)!!!$$

In the case of continuous distribution in its simplest form: $\mu(\beta) \equiv 1$.

$$g_1(s) = \frac{s-1}{\log s}, \quad g_2(s) = \frac{s \log s}{s-1}.$$

Properties of the kernels

Theorem. Let $\mu(\beta)$ be either of the form (3) or of the form (4) with the additional assumptions $\mu \in C^3[0,1]$, $\mu(1) \neq 0$, and $\mu(0) \neq 0$ or $\mu(\beta) = a\beta^\nu$ as $\beta \to 0$, where $a, \nu > 0$. Then for $i = 1, 2,$:

(a) $k_i \in L^1_{loc}(\mathbb{R}_+)$ and $\lim_{t\to+\infty} k_i(t) = 0$;

(b) $k_i(t) \in \mathcal{CMF}$ for $t > 0$;

(c) $k_1 * k_2 \equiv 1$;

(d) $g_i(s) \in \mathcal{BF}$ for $s > 0$;

(e) $g_i(s)/s \in \mathcal{CMF}$ for $s > 0$;

(f) $g_i(s)$ admits analytic extension to the sector $|\arg s| < \pi$ and

$$|\arg g_i(s)| \leq |\arg s|, \quad |\arg s| < \pi.$$

In the discrete distribution case a stronger inequality holds:

$$|\arg g_i(s)| \leq \alpha |\arg s|, \quad |\arg s| < \pi.$$

The classical abstract Cauchy problem:

$$u'(t) = Au(t), \quad t > 0; \qquad u(0) = a \in X.$$

Main result:

Assume that the classical Cauchy problem is well-posed with solution $u(t)$ satisfying

$$\|u(t)\| \leq M\|a\|, \quad t \geq 0.$$

Then any of the problems

$$\int_0^1 \mu(\beta)\, {}^C D_t^\beta u(t)\, d\beta = Au(t), \quad t > 0, \qquad u(0) = a \in X,$$

$$u'(t) = \int_0^1 \mu(\beta) D_t^\beta A u(t)\, d\beta, \quad t > 0, \qquad u(0) = a \in X$$

is well-posed with solution satisfying the same estimate.

The classical abstract Cauchy problem:

$$u'(t) = Au(t), \quad t > 0; \qquad u(0) = a \in X.$$

$T(t)$ - solution operator (defined by $T(t)a = u(t),\ t \geq 0$);

$R(s, A)$ - resolvent operator:

$$R(s, A) = (s - A)^{-1} = \int_0^\infty e^{-st} T(t)\, dt, \quad s > 0,$$

The Hille-Yosida theorem states that the classical Cauchy problem is well-posed with solution operator $T(t)$ such that $\|T(t)\| \leq M, \quad t \geq 0$, iff $R(s, A)$ is well defined for $s \in (0, \infty)$ and

$$\|R(s, A)^n\| \leq \frac{M}{s^n}, \quad s > 0,\ n \in \mathbb{N}.$$

Abstract Volterra integral equation

$$u(t) = a + \int_0^t k(t-\tau) A u(\tau)\, d\tau, \quad t \geq 0; \qquad a \in X,$$

The Laplace transform of the solution operator $S(t)$

$$H(s) = \int_0^\infty e^{-st} S(t)\, dt, \quad s > 0$$

is given by

$$H(s) = \frac{g(s)}{s} R(g(s), A), \quad g(s) = 1/\widehat{k}(s).$$

The Generation Theorem (Prüss, 1993) states that the integral equation is well-posed with solution operator $S(t)$ satisfying $\|S(t)\| \leq M,\ t \geq 0$, iff

$$\|H^{(n)}(s)\| \leq M \frac{n!}{s^{n+1}}, \quad \text{for all } s > 0,\ n \in \mathbb{N}_0.$$

Main result

Theorem.
Suppose that the classical Cauchy problem is well-posed with solution $u(t)$ satisfying

$$\|u(t)\| \leq M\|a\|, \quad t \geq 0.$$

Then problems (1) and (2) are well-posed and their solutions satisfy the same estimate.

Proof:
We know

$$\|R(s,A)^n\| \leq M/s^n, \quad s > 0, \ n \in \mathbb{N}.$$

We have to prove

$$\|H^{(n)}(s)\| \leq M\frac{n!}{s^{n+1}}, \quad \text{for all } s > 0, \ n \in \mathbb{N}_0,$$

where

$$H(s) = \frac{g(s)}{s}R(g(s),A),$$

and $g(s) = 1/\widehat{k}(s)$, $R(s,A) = (s-A)^{-1}$.

By the Leibniz rule:

$$H^{(n)}(s) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{g(s)}{s}\right)^{(n-k)} w^{(k)}(s),\ w(s) = R(g(s), A). \tag{8}$$

Formula for the k-th derivative of a composite function:

$$w^{(k)}(s) = \sum_{p=1}^{k} a_{k,p}(s)(-1)^p p!(R(g(s), A))^{p+1}, \tag{9}$$

where the functions $a_{k,p}(s)$ are defined by

$$\begin{aligned} & a_{k+1,p}(s) = a_{k,p-1}(s)g'(s) + a'_{k,p}(s), \quad 1 \le p \le k+1,\ k \ge 1, \\ & a_{k,0} = a_{k,k+1} \equiv 0, \quad a_{1,1}(s) = g'(s). \end{aligned} \tag{10}$$

$$g(s) \in \mathcal{BF} \Rightarrow (-1)^{k+p} a_{k,p}(s) \in \mathcal{CMF}. \tag{11}$$

Proof: by induction.

So far:

$$(-1)^n H^{(n)}(s) = \sum_{k=0}^{n} \sum_{p=1}^{k} b_{n,k,p}(s)(R(g(s), A))^{p+1} \tag{12}$$

where

$$b_{n,k,p}(s) = (-1)^{n+p} \binom{n}{k} \left(\frac{g(s)}{s}\right)^{(n-k)} a_{k,p}(s)p!$$

Positivity?

$$(-1)^{k+p} a_{k,p}(s) \geq 0, \quad g(s) \in \mathcal{BF} \Rightarrow g(s)/s \in \mathcal{CMF}, \quad s > 0. \tag{13}$$

$$\begin{aligned} \Rightarrow b_{n,k,p}(s) &= (-1)^{n+p} \binom{n}{k} \left(\frac{g(s)}{s}\right)^{(n-k)} a_{k,p}(s)p! \\ &= \binom{n}{k} (-1)^{n-k} \left(\frac{g(s)}{s}\right)^{(n-k)} (-1)^{k+p} a_{k,p}(s)p! \geq 0 \end{aligned}$$

$$(-1)^n H^{(n)}(s) = \sum_{k=0}^{n} \sum_{p=1}^{k} b_{n,k,p}(s)(R(g(s), A))^{p+1}$$

$$
\begin{aligned}
\Rightarrow \|H^{(n)}(s)\| &\leq \sum_{k=0}^{n}\sum_{p=1}^{k} b_{n,k,p}(s)\|(R(g(s),A))^{p+1}\| \\
&\leq M\sum_{k=0}^{n}\sum_{p=1}^{k} b_{n,k,p}(s)((g(s))^{-(p+1)} \\
&= M(-1)^n(s^{-1})^{(n)} = Mn!s^{-(n+1)}, \quad s>0.
\end{aligned}
$$

where we have used that for $A \equiv 0$:

$$
(-1)^n(s^{-1})^{(n)} = \sum_{k=0}^{n}\sum_{p=1}^{k} b_{n,k,p}(s)(g(s))^{-(p+1)}.
$$

Therefore, the conditions of the Generation Theorem are satisfied and the problems are well-posed with bounded solution operators $S(t)$, satisfying $\|S(t)\| \leq M$, $t \geq 0$.

Subordination formula

$T(t)$ - the solution operator of the classical Cauchy problem.
Under the assumptions of the previous theorem, the solution operator $S(t)$ of problem (1), resp. (2), satisfies the subordination identity

$$S(t) = \int_0^\infty \varphi(t,\tau) T(\tau)\, d\tau, \quad t > 0, \tag{14}$$

with function $\varphi(t,\tau)$ defined by

$$\varphi(t,\tau) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma - \mathrm{i}\infty}^{\gamma + \mathrm{i}\infty} e^{st - \tau g(s)} \frac{g(s)}{s}\, ds, \quad \gamma, t, \tau > 0, \tag{15}$$

The function $\varphi(t,\tau)$ is a probability density function, i.e. it satisfies the properties

$$\varphi(t,\tau) \geq 0, \quad \int_0^\infty \varphi(t,\tau)\, d\tau = 1. \tag{16}$$

Hint: take function $\varphi(t,\tau)$ such that $\mathcal{L}_t\{\varphi\}(s,\tau) = \frac{g(s)}{s} e^{-\tau g(s)}, \quad s, \tau > 0.$

Conclusions

Various possibilities for the operator A: e.g. the Laplace operator, general second order symmetric uniformly elliptic operators, operators leading to the so-called time-space fractional equations, such as: space-fractional derivatives (e.g. in the Riesz sense), fractional powers of the multi-dimensional Laplace operator, other forms of fractional Laplacian, fractional powers of more general elliptic operators, etc.

The developed technique is applicable to more general abstract Volterra integral equations with kernel $k(t)$, which Laplace transform $\widehat{k}(s)$ is well-defined for $s > 0$ and is such that $(\widehat{k}(s))^{-1}$ is a Bernstein function.

Acknowledgments

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